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Geometric discretization of the Bianchi system

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Abstract

We introduce the dual Koenigs lattices, which are the integrable discrete analogues of conjugate nets with equal tangential invariants, and we find the corresponding reduction of the fundamental transformation. We also introduce the notion of discrete normal congruences. Finally, considering quadrilateral lattices "with equal tangential invariants" which allow for harmonic normal congruences we obtain, in complete analogy with the continuous case, the integrable discrete analogue of the Bianchi system together with its geometric meaning. To obtain this geometric meaning we also make use of the novel characterization of the circular lattice as a quadrilateral lattice whose coordinate lines intersect orthogonally in the mean.

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1. Introduction

The paper concerns with the integrable discrete analogue of the following nonlinear partial differential equation

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$$\vec{\boldsymbol{N}}_{,uv} = -\frac{\vec{\boldsymbol{N}}_{,u}\cdot\vec{\boldsymbol{N}}_{,v}}{U+V}\vec{\boldsymbol{N}}, \qquad \vec{\boldsymbol{N}}\cdot\vec{\boldsymbol{N}} = U+V, \tag{1}$$

where $\vec{\boldsymbol{N}}$ is a vector valued function of two variables u and v, comma denotes differentiation (e.g., $\vec{\boldsymbol{N}}_{,u} = \partial\vec{\boldsymbol{N}}/\partial u$), and $U = U(u)$ and $V = V(v)$ are given functions of single arguments, u and v, respectively. The system (1) was introduced long time ago by Bianchi [3] as an equation satisfied by a normal vector of special hyperbolic surfaces in $\mathbb{E}^3$; here (u, v) are the asymptotic coordinates of the surface. The second geometric interpretation of the Bianchi system is given in terms of conjugate nets in $\mathbb{E}^3$ permanent in deformation [17]. Such nets can be equivalently characterized as conjugate nets with equal tangential invariants allowing for harmonic normal congruences [18].

The third and more recent geometric interpretation of the Bianchi system follows from its equivalence (with the signature of the scalar product changed to $+--$) to the Ernst-like reduction of Einstein's equation describing the interaction of gravitational waves [6,20]. Indeed, define [28]

$$\xi = \frac{N_1 + \mathrm{i}N_2}{\sqrt{r}+N_0}, \quad r = \vec{\boldsymbol{N}}\cdot\vec{\boldsymbol{N}} = N_0^2 + \epsilon(N_1^2+N_2^2), \ \ \epsilon = \pm 1;$$

then the Bianchi system is transformed into the equations

$$\left(2\xi_{,uv} + \frac{r_{,v}}{r}\xi_{,u} + \frac{r_{,u}}{r}\xi_{,v}\right)(\xi\bar{\xi}+\epsilon) = 4\bar{\xi}\xi_{,u}\xi_{,v}, \quad r_{,uv} = 0,$$

which are the hyperbolic version of the Ernst system describing axisymmetric stationary vacuum solutions of the Einstein equations [2,19,25,27].

During the last few years the integrable discrete (difference) analogues of geometrically significant integrable differential equations have attracted considerable attention [5,14,16,23]. The integrable discrete systems appear naturally in statistical physics [21] and in quantum field theory [24]. It has long been expected that a quantization of gravitational systems will lead to a discrete structure of space–time and then the differential equations must be replaced by difference equations at a fundamental level (see recent reviews [1,22]). It would be therefore useful and instructive to study the geometric properties of integrable discrete version of the Bianchi system, a distinctive integrable reduction of Einstein's equations.

In a recent work [13,28], in which the original paper [3] of Bianchi was discretized step by step, the following nonlinear vector equation

$$\vec{\boldsymbol{N}}_{(12)} + \vec{\boldsymbol{N}} = \frac{U_1+U_2}{(\vec{\boldsymbol{N}}_{(1)}+\vec{\boldsymbol{N}}_{(2)})(\vec{\boldsymbol{N}}_{(1)}+\vec{\boldsymbol{N}}_{(2)})}(\vec{\boldsymbol{N}}_{(1)}+\vec{\boldsymbol{N}}_{(2)}) \tag{2}$$

was derived, in the context of asymptotic lattices, and identified as the proper integrable discrete analogue of the Bianchi system (1). In Eq. (2), $U_1 = U_1(m_1)$ and $U_2 = U_2(m_2)$ are given functions, respectively, of the single arguments m_1 and m_2, and the subscripts in bracket denote shifts in the indexed variables, i.e., for $m_1, m_2 \in \mathbb{Z}^2$ we write $\vec{\boldsymbol{N}}_{(\pm 1)}(m_1, m_2) = \vec{\boldsymbol{N}}(m_1 \pm 1, m_2)$, $\vec{\boldsymbol{N}}_{(\pm 2)}(m_1, m_2) = \vec{\boldsymbol{N}}(m_1, m_2 \pm 1)$, $\vec{\boldsymbol{N}}_{(\pm 1 \pm 2)}(m_1, m_2) = \vec{\boldsymbol{N}}(m_1 \pm 1, m_2 \pm 1)$.

It is remarkable that the same system (2) had been already introduced, a bit earlier, in [33], in the different geometric context of discrete isothermic nets, as an integrable discretization, instead, of the vectorial Calapso equation. Therefore, the system (2) can be viewed as a remarkable example in which discretization leads to a unification of different geometries and of different partial differential equations.

Eq. (2) can be obtained from the discrete Moutard equation found in [29]

$$\vec{\boldsymbol{N}}_{(12)} + \vec{\boldsymbol{N}} = F(\vec{\boldsymbol{N}}_{(1)} + \vec{\boldsymbol{N}}_{(2)}), \tag{3}$$

imposing the integrable quadratic constraint

$$(\vec{\boldsymbol{N}}_{(12)} + \vec{\boldsymbol{N}})(\vec{\boldsymbol{N}}_{(1)} + \vec{\boldsymbol{N}}_{(2)}) = U_1 + U_2. \tag{4}$$

The integrability of the discrete Bianchi system (2) was proven in [13,33] using different approaches, by showing the compatibility of the constraint (4) with a suitable restriction of the Darboux transformation of the discrete Moutard equation (3). However, the geometric meaning of the constraint (4) was still unclear both in the context of discrete isothermic nets and in the context of discrete asymptotic nets. This situation seemed to be in contrast to the believe that geometry, and especially the discrete geometry, should help to understand integrability of the underlying systems.

In the present paper we obtain the integrable discrete analogue of the Bianchi system (1) in a geometric way. In order to keep track of integrability on each step of the construction, we make use of its geometric interpretation as the system describing *conjugate nets with equal tangential invariants allowing for harmonic normal congruences* [18].

The main results of the paper can be stated as follows: (i) we have introduced the notion of *dual Koenigs lattice*, the proper integrable discrete analogue of a conjugate net with equal tangential invariants; (ii) we have introduced the notion of *discrete normal congruence*, the proper integrable discrete analogue of a normal congruence; (iii) we have introduced the (quadrilateral) Bianchi lattice as a *dual Koenigs lattice allowing for a harmonic congruence which is a discrete normal congruence*. Because the simultaneous application of several integrable constraints preserves integrability, we obtain for free the integrability of the Bianchi lattice. An additional result of the paper, relevant in deriving the above results, is the novel geometric characterization of the circular lattice as a quadrilateral lattice whose coordinate lines intersect orthogonally in the mean.

It turns out that applying the above geometric constraints to the quadrilateral lattice one obtains

$$\vec{\boldsymbol{n}}_{(12)} + \vec{\boldsymbol{n}} = F_{(1)}\vec{\boldsymbol{n}}_{(1)} + F_{(2)}\vec{\boldsymbol{n}}_{(2)}, \qquad \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{n}} F = U_1 + U_2, \tag{5}$$

which is simply related to Eqs. (3) and (4) by

$$\vec{\boldsymbol{n}} = \vec{\boldsymbol{N}}_{(1)} + \vec{\boldsymbol{N}}_{(2)}.$$

In fact, the system (5) appeared first in a hidden form in [28] (see also [13]) as a useful tool to prove the integrability of Eq. (2). The program realized in this paper allows to embed yet another integrable discrete system into the general theory of lattices with planar quadrilaterals [7,8,12,14,16,23], which are the proper discrete analogues of conjugate nets. Moreover, the paper confirms once more that, in the context of integrable geometries, the transition *from differential to discrete*, which is often highly nontrivial on the level of the direct calculations, can be simplified and better understood on geometric level.

The paper is constructed as follows. We start from Section 2, in which we present the integrable discrete analogues of nets with equal tangential invariants, combining the notion of dual lattice [15] with the notion of Koenigs lattice [11]. In Section 3 we introduce and study the discrete normal congruences. Finally, in Section 4 we put all the above ingredients together to get the integrable discrete Bianchi system. Appendices A and B contains additional results of more algebraic nature stated in the language of partial difference equations. In Appendix A, we present the algebraic version of the geometric characterization of circular lattices and normal congruences obtained in Section 3. Appendix B contains the Darboux transformation for the discrete Bianchi system (5) written as a linear problem.

We remark that, during the conference in which the geometric discretization described in this paper was presented, also the discretization of the notion of conjugate net invariant for deformation was presented [34]. Therefore, two of the geometric meanings of the Bianchi system (1) seem to be now successfully discretized.

2. The dual Koenigs lattices

As it was mentioned above, in the geometric derivation of the Bianchi system (1) one combines the notion of the net with equal tangential invariants with the notion of the normal congruence. This section is devoted to the presentation of the integrable discrete analogue of the first component of the geometric definition of the Bianchi system. We start with collecting some by now classical results from the theory of two-dimensional quadrilateral lattices in three-dimensional space [8,14,32], together with their tangential description [15], and from some recent results on the theory of Koenigs lattices (the discrete analogues of conjugate nets with equal point invariants) [11].

2.1. Quadrilateral lattices and discrete congruences

The integrable discrete analogues of conjugate nets are lattices made out of planar quadrilaterals [8,32], called in [14] the quadrilateral lattices. Given such a lattice $x : \mathbb{Z}^2 \to \mathbb{P}^3$, then, in terms of the homogeneous coordinates $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{R}^4_*$ of the lattice, the planarity of the elementary quadrilaterals can be expressed as a linear relation between $\boldsymbol{x}$, $\boldsymbol{x}_{(1)}$, $\boldsymbol{x}_{(2)}$ and $\boldsymbol{x}_{(12)}$. In the generic situation, assumed in the sequel, in which three vertices of the elementary quadrilaterals are never collinear, the linear relation can be written down as the discrete Laplace equation [8,14]

$$\boldsymbol{x}_{(12)} = A_{(1)}\boldsymbol{x}_{(1)} + B_{(2)}\boldsymbol{x}_{(2)} + C\boldsymbol{x}, \tag{6}$$

where A, B and C are real functions on $\mathbb{Z}^2$.

Any quadrilateral lattice in $\mathbb{P}^3$ can be considered as envelope of its tangent planes (the planes of elementary quadrilaterals). Denote by $\boldsymbol{x}^* \in \mathbb{R}^4_*$ the homogeneous coordinates of the plane $x^* \in (\mathbb{P}^3)^*$ passing through x, $x_{(1)}$ and $x_{(2)}$ (see Fig. 1), i.e.,

$$\langle \boldsymbol{x}^*, \boldsymbol{x} \rangle = \langle \boldsymbol{x}^*, \boldsymbol{x}_{(1)} \rangle = \langle \boldsymbol{x}^*, \boldsymbol{x}_{(2)} \rangle = 0.$$

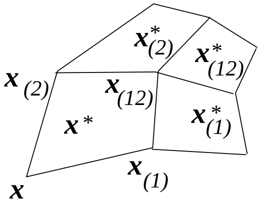

Fig. 1. The quadrilateral lattice.

Because the planes x^*, $x^*_{(1)}$, $x^*_{(2)}$ and $x^*_{(12)}$ intersect in the point $x_{(12)}$, then also the homogeneous coordinates $\boldsymbol{x}^* : \mathbb{Z}^2 \to \mathbb{R}^4_*$ satisfy the Laplace equation

$$\boldsymbol{x}^*_{(12)} = A^*_{(1)}\boldsymbol{x}^*_{(1)} + B^*_{(1)}\boldsymbol{x}^*_{(2)} + C^*\boldsymbol{x}^*. \tag{7}$$

The homogeneous coordinates $\boldsymbol{x}$ are called the point coordinates of the lattice x while $\boldsymbol{x}^*$ are called the tangential coordinates of the lattice (see also [15]).

The theory of transformations of the quadrilateral lattices is based on the theory of congruences of lines [16]. A $\mathbb{Z}^2$ family of straight lines L in $\mathbb{P}^3$ is called *congruence* if any two neighboring lines are coplanar (and therefore intersect). The intersection points $y_i = L \cap L_{(-i)}$, $i = 1, 2$, define focal lattices of the congruence. It turns out that the focal lattices have planar quadrilaterals as well (see Fig. 2).

Corollary 1. *Notice, that the plane y^*_1 of the first focal lattice contains the lines L and $L_{(2)}$, while the point y_1 is the intersection of the lines L and $L_{(-1)}$. This implies that one cannot just "dualize" formulas where focal lattices appear. In general, in place of y_1 it should be written $y^*_{2(-1)}$ while in place of y_2 it should be written $y^*_{1(-2)}$, see* Fig. 2.

A quadrilateral lattice and a congruence are said to be *conjugate* to each other if the points of the lattice belong to the corresponding lines of the congruence. A quadrilateral lattice and a congruence are said to be *harmonic* to each other if the lines of the congruence belong to the corresponding tangent planes of the lattice. Under the standard dualization in the projective space $\mathbb{P}^3$ (see e.g. [31]), the statement "the point belongs to the line" is replaced by the statement "the plane contains the line". This implies that the notions of conjugate and harmonic congruences (to a quadrilateral lattice) are dual to each other.

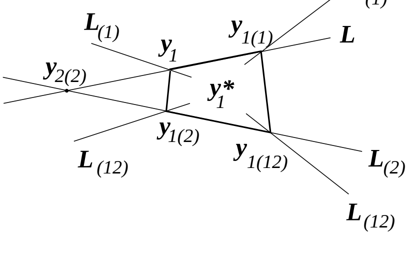

Fig. 2. The first focal lattice of a congruence.

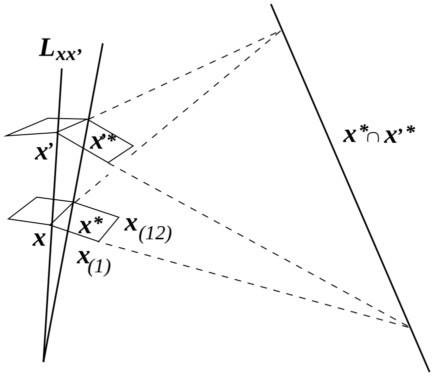

Fig. 3. The fundamental transformation.

The quadrilateral lattice x' is a *fundamental transform* of x if both lattices are conjugate to the same congruence [16] (see Fig. 3), called the conjugate congruence of the transformation.

It turns out that both lattices x and x' are also harmonic to the congruence given by the intersection of the planes x^* and x'^*, which is called the harmonic congruence of the transformation. This makes the geometric picture of the fundamental transformation in $\mathbb{P}^3$ self-dual.

2.2. *The dual Koenigs lattice*

We first recall the geometric definition of a Koenigs lattice and its algebraic characterization introduced in [11]. Given a quadrilateral lattice x in $\mathbb{P}^3$, denote by x_1 the intersection points of the lines $L_{xx_{(2)}}$ with the lines $L_{x_{(-1)}x_{(-12)}}$ and denote by x_{-1} the intersection points of the lines $L_{xx_{(1)}}$ with the lines $L_{x_{(-2)}x_{(1-2)}}$.

Definition 1. *The Koenigs lattice* is a two-dimensional quadrilateral lattice such that, for any point x of the lattice, there exists a conic passing through the six points $x_1, x_{1(1)}, x_{1(11)}$, $x_{-1}, x_{-1(2)}$ and $x_{-1(22)}$.

Proposition 1. *The Laplace equation of the Koenigs lattice can be gauged into the canonical form*

$$\boldsymbol{x}_{(12)} + \boldsymbol{x} = F_{(1)}\boldsymbol{x}_{(1)} + F_{(2)}\boldsymbol{x}_{(2)}. \tag{8}$$

Using Pascal's "mystic hexagon" theorem (see e.g. [31]), it is possible to obtain the following alternative geometric characterization of a Koenigs lattice (see Fig. 4).

Proposition 2. *The quadrilateral lattice x in $\mathbb{P}^3$ is a Koenigs lattice if and only if, for any point x of the lattice, the three lines*

(i) *the line connecting the points $x^* \cap x^*_{(-1)} \cap x^*_{(-1-1)}$ and $x^* \cap x^*_{(2)} \cap x^*_{(22)}$,*
(ii) *the line connecting the points $x^* \cap x^*_{(-2)} \cap x^*_{(-2-2)}$ and $x^* \cap x^*_{(1)} \cap x^*_{(11)}$,*

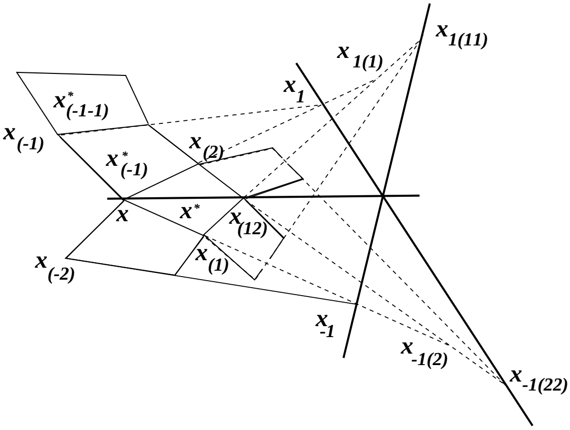

Fig. 4. The three lines in the definition of the Koenigs lattice.

(iii) *the line connecting the points* $x^* \cap x^*_{(-1)} \cap x^*_{(-2)}$ *and* $x^* \cap x^*_{(1)} \cap x^*_{(2)}$, *intersect in a single point*.

Proof. By the Pascal theorem, the quadrilateral lattice x is a Koenigs lattice if and only if the lines $L_{x_{-1}x_{1(11)}}$ and $L_{x_1x_{-1(22)}}$ intersect in a point of $L_{xx_{(12)}}$. Notice that in $\mathbb{P}^3$ the point x_1 is the intersection point of the three planes x^*, $x^*_{(-1)}$ and $x^*_{(-1-1)}$. Similarly, the point x_{-1} is the intersection point of three planes x^*, $x^*_{(-2)}$ and $x^*_{(-2-2)}$. By definition, in $\mathbb{P}^3$ we have $x = x^* \cap x^*_{(-1)} \cap x^*_{(-2)}$ and $x_{(12)} = x^* \cap x^*_{(1)} \cap x^*_{(2)}$. □

Remark. For a generic quadrilateral lattice x in $\mathbb{P}^3$, the three lines of the above proposition are contained in the plane x^*.

The notion of the dual Koenigs lattice can be conveniently obtained by dualizing the geometric definition of the Koenigs lattice given in the above proposition.

Definition 2. A quadrilateral lattice in $\mathbb{P}^3$ is called the *dual Koenigs lattice* if, for any point x of the lattice, the three lines

(i) the intersection line of the plane $\pi_{xx_{(-1)}x_{(-1-1)}}$ with the plane $\pi_{xx_{(2)}x_{(22)}}$,
(ii) the intersection line of the plane $\pi_{xx_{(-2)}x_{(-2-2)}}$ with the plane $\pi_{xx_{(1)}x_{(11)}}$,
(iii) the intersection line of the plane $\pi_{xx_{(-1)}x_{(-2)}}$ with the plane $\pi_{xx_{(1)}x_{(2)}}$, are coplanar.

Remark. For a generic quadrilateral lattice in $\mathbb{P}^3$ the three lines of the above definition intersect in the point x.

Also the basic algebraic characterization of the dual Koenigs lattice can be obtained dualizing the corresponding algebraic characterization of the Koenigs lattice (just replacing the homogeneous point coordinates by the tangential ones). Because of the importance of this characterization in the paper, we will prove it without referring to the duality principle.

Proposition 3. *The quadrilateral lattice x in $\mathbb{P}^3$ is a dual Koenigs lattice if and only if its tangential coordinates can be gauged in such a way that the Laplace equation (7) takes the form*

$$\boldsymbol{x}^*_{(12)} + \boldsymbol{x}^* = F_{(1)}\boldsymbol{x}^*_{(1)} + F_{(2)}\boldsymbol{x}^*_{(2)}. \tag{9}$$

Proof. The plane $\pi_{xx_{(1)}x_{(11)}}$ contains the lines $L_{x\,x_{(1)}} = x^* \cap x^*_{(-2)}$ and $L_{x_{(1)}\,x_{(11)}} = x^*_{(1)} \cap x^*_{(1-2)}$. Therefore, the homogeneous coordinates of the plane can be found by solving the linear system

$$\lambda\boldsymbol{x}^* + \mu\boldsymbol{x}^*_{(-2)} = \sigma\boldsymbol{x}^*_{(1)} + \delta\boldsymbol{x}^*_{(1-2)}.$$

Using the Laplace equation (7) it can be shown that the coordinates read

$$B^*\boldsymbol{x}^* + C^*_{(-2)}\boldsymbol{x}^*_{(-2)} = \boldsymbol{x}^*_{(1)} - A^*_{(1-2)}\boldsymbol{x}^*_{(1-2)}.$$

Similarly, the homogeneous coordinates of the planes $\pi_{xx_{(2)}x_{(22)}}$, $\pi_{xx_{(-1)}x_{(-1-1)}}$ and $\pi_{xx_{(-2)}x_{(-2-2)}}$ are given, correspondingly, by

$$\begin{aligned}
&A^*\boldsymbol{x}^* + C^*_{(-1)}\boldsymbol{x}^*_{(-1)} = \boldsymbol{x}^*_{(2)} - B^*_{(-12)}\boldsymbol{x}^*_{(-12)},\\
&B^*_{(-1-1)}\boldsymbol{x}^*_{(-1-1)} + C^*_{(-1-1-2)}\boldsymbol{x}^*_{(-1-1-2)} = \boldsymbol{x}^*_{(-1)} - A^*_{(-1-2)}\boldsymbol{x}^*_{(-1-2)},\\
&A^*_{(-2-2)}\boldsymbol{x}^*_{(-2-2)} + C^*_{(-1-2-2)}\boldsymbol{x}^*_{(-1-2-2)} = \boldsymbol{x}^*_{(-2)} - B^*_{(-1-2)}\boldsymbol{x}^*_{(-1-2)}.
\end{aligned}$$

The coordinates of the planes $\pi_{xx_{(1)}x_{(2)}}$ and $\pi_{xx_{(-1)}x_{(-2)}}$ are, by definition, $\boldsymbol{x}^*$ and $\boldsymbol{x}^*_{(-1-2)}$. There exists a plane containing the intersection lines $\pi_{xx_{(1)}x_{(11)}} \cap \pi_{xx_{(-2)}x_{(-2-2)}}$, $\pi_{xx_{(2)}x_{(22)}} \cap \pi_{xx_{(-1)}x_{(-1-1)}}$ and $\pi_{xx_{(1)}x_{(2)}} \cap \pi_{xx_{(-1)}x_{(-2)}}$ if and only if the linear system for the unknowns $\lambda, \mu, \sigma, \delta, \chi, \nu$

$$\begin{aligned}
&\lambda(B^*\boldsymbol{x}^* + C^*_{(-2)}\boldsymbol{x}^*_{(-2)}) + \mu(\boldsymbol{x}^*_{(-2)} - B^*_{(-1-2)}\boldsymbol{x}^*_{(-1-2)})\\
&\quad = \sigma(A^*\boldsymbol{x}^* + C^*_{(-1)}\boldsymbol{x}^*_{(-1)}) + \delta(\boldsymbol{x}^*_{(-1)} - A^*_{(-1-2)}\boldsymbol{x}^*_{(-1-2)}) = \chi\boldsymbol{x}^* + \nu\boldsymbol{x}^*_{(-1-2)}
\end{aligned}$$

has a nontrivial solution. By linear algebra, and assuming that no three of the four points x, $x_{(-1)}$, $x_{(-2)}$ and $x_{(-1-2)}$ are collinear, such a solution exists when

$$A^*C^*_{(-2)}B^*_{(-1-2)} = B^*C^*_{(-1)}A^*_{(-1-2)}.$$

This restriction on the coefficients of the Laplace equation (7) implies, exactly like in the corresponding proposition of [11], existence of the gauge function ρ defined by

$$\rho_{(12)} = -C^*\rho, \qquad \rho_{(1)}A^* = \rho_{(2)}B^*.$$

After the gauge transformation $\boldsymbol{x}^* \mapsto \boldsymbol{x}^*/\rho$, the new homogeneous tangential coordinates of the lattice satisfy the Laplace equation of the form (9) with the potential

$$F = \frac{A^*\rho}{\rho_{(2)}} = \frac{B^*\rho}{\rho_{(1)}}. \qquad \square$$

After having established the relation between Koenigs lattices and dual Koenigs lattices on both geometric and algebraic levels, we will make use of the results of [11] to present the dual version of other basic properties of the Koenigs lattices. Taking into account the exchange of indexes and shifts in the notation for the tangential coordinates of the focal lattices of the harmonic congruence (see Corollary 1), we will obtain the geometric characterization of the dual Koenigs lattices from the point of view of their transformations.

Consider a quadrilateral lattice x in $\mathbb{P}^3$ and a congruence L harmonic to the lattice. According to the dual description of lines, any line of the congruence is identified with the pencil of planes containing the line. Notice that the planes z_1^*, z_2^*, $z_{1(-2)}^*$ and $z_{2(-1)}^*$ of the focal lattices of the congruence, as well as the plane x^*, are elements of the same pencil. On each line there exists a unique involution, denoted by $\mathfrak{i}$, which maps the planes z_1^* and z_2^* to the planes $z_{1(-2)}^*$ and $z_{2(-1)}^*$, correspondingly. The following proposition is the dual version of Corollary 15 of [11].

Proposition 4. *The quadrilateral lattice x in $\mathbb{P}^3$ is a dual Koenigs lattice if and only if, for any congruence harmonic to the lattice, the planes $x'^* = \mathfrak{i}(x^*)$, where $\mathfrak{i}$ are the involutions described above, are tangent planes of a quadrilateral lattice.*

Corollary 2. *If x is a dual Koenigs lattice, then the quadrilateral lattice x'* with tangent planes $x'^* = \mathfrak{i}(x^*)$ *must be, by the symmetry of the construction, a dual Koenigs lattice as well.*

The above proposition provides the basic geometric characterization of the reduction of the fundamental transformation corresponding to the dual Koenigs lattices. Its algebraic formulation, given by the dual version of Proposition 8 of [11], reads as follows: here Δ_i, $i = 1, 2$, denote the standard partial difference operators acting as $\Delta_i f = f_{(i)} - f$.

Theorem 1. *Given a dual Koenigs lattice with tangential coordinates $\boldsymbol{x}^*$ in the canonical gauge of equation (9). If θ is a scalar solution of the discrete Moutard equation*

$$\theta_{(12)} + \theta = F(\theta_{(1)} + \theta_{(2)}), \tag{10}$$

then the solution $\boldsymbol{x}'^*$ of the linear system

$$\Delta_1\left(\frac{\boldsymbol{x}'^*}{\phi'}\right) = \theta_{(1)}\theta_{(12)}\Delta_1\left(\frac{\boldsymbol{x}^*}{\phi}\right), \tag{11}$$

$$\Delta_2\left(\frac{\boldsymbol{x}'^*}{\phi'}\right) = -\theta_{(2)}\theta_{(12)}\Delta_2\left(\frac{\boldsymbol{x}^*}{\phi}\right) \tag{12}$$

with ϕ and ϕ' given by

$$\phi = \theta_{(1)} + \theta_{(2)}, \qquad \phi' = \frac{1}{\theta_{(1)}} + \frac{1}{\theta_{(2)}}, \tag{13}$$

gives the tangential coordinates of a new dual Koenigs lattice satisfying equation (9), with the potential

$$F' = F\frac{\theta_{(1)}\theta_{(2)}}{\theta\theta_{(12)}}. \tag{14}$$

Finally, we collect the dual versions of some formulas scattered in [11] which turn out to be useful in the sequel.

Corollary 3. *The homogeneous coordinates of the tangent planes of the focal lattices* z_i, $i = 1, 2$ *of the harmonic congruence of the dual discrete Koenigs transformation are given by*

$$z_1^* = -\left(\theta\theta_{(1)}\frac{\boldsymbol{x}^*}{\phi} + \frac{\boldsymbol{x}'^*}{\phi'}\right)_{(2)} = -\theta_{(2)}\theta_{(12)}\frac{\boldsymbol{x}^*}{\phi} - \frac{\boldsymbol{x}'^*}{\phi'}, \tag{15}$$

$$z_2^* = \left(\theta\theta_{(2)}\frac{\boldsymbol{x}^*}{\phi} - \frac{\boldsymbol{x}'^*}{\phi'}\right)_{(1)} = \theta_{(1)}\theta_{(12)}\frac{\boldsymbol{x}^*}{\phi} - \frac{\boldsymbol{x}'^*}{\phi'}. \tag{16}$$

They can be found from equations

$$z^*_{2(-1)} - z^*_{1(-2)} = \theta\boldsymbol{x}^*, \tag{17}$$

$$\Delta_1 z^*_{1(-2)} = -\theta_{(1)}(\boldsymbol{x}^*_{(1)} - F\boldsymbol{x}^*), \tag{18}$$

$$\Delta_2 z^*_{2(-1)} = \theta_{(2)}(\boldsymbol{x}^*_{(2)} - F\boldsymbol{x}^*), \tag{19}$$

once the solution θ *of* Eq. (10) *is given. Moreover, the homogeneous coordinates of the fixed planes of the involutions* $\mathfrak{i}$, *are given by*

$$z^*_{\pm} = \pm\sqrt{\theta\theta_{(12)}}\boldsymbol{x}^* + \sqrt{\theta_{(1)}\theta_{(2)}}\boldsymbol{x}'^*. \tag{20}$$

3. The discrete normal congruences

The classical examples [17] of congruences are given by normals to a surface in the three-dimensional Euclidean space $\mathbb{E}^3$. The developables of such congruences cut the surface along the curvature lines and the corresponding tangent planes of its focal nets intersect orthogonally. Keeping the last property one obtains the notion of the normal congruence. In this section we will define the discrete analogue of such congruences.

The discrete analogues of surfaces parameterized by curvature lines were introduced in [26,30] and are two-dimensional lattices in $\mathbb{E}^3$ made of circular quadrilaterals. Integrability of multidimensional circular lattices was proven first geometrically in [7] (see also [4]), and then by using analytic means in [12].

Let us present a new elementary geometric characterization of circular lattices which will be important in the characterization of the Bianchi lattices (for other algebraic characterizations of circular lattices[7,9,23] see Appendices A and B).

Proposition 5. *A planar quadrilateral is circular if and only if the lines bisecting the angles between its opposite sides and intersecting the quadrilateral are orthogonal* (*see* Fig. 5).

Proof. By elementary geometry, using the following facts: (i) a convex planar quadrilateral is circular if and only if its opposite angles add to π; (ii) a nonconvex planar quadrilateral is circular if and only if its opposite angles are equal. □

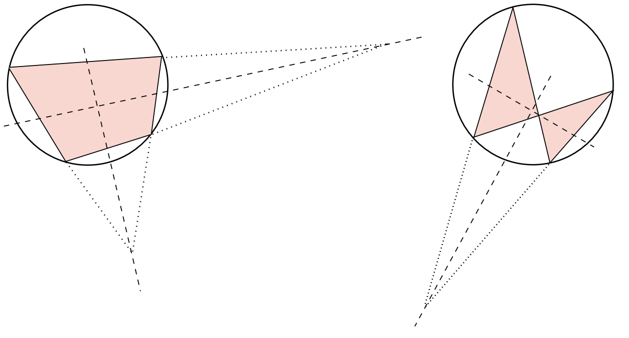

Fig. 5. Circular quadrilaterals.

Remark. The above result means that the coordinate lines of the circular lattice intersect orthogonally "in the mean".

In [10] it was shown geometrically that the normals to the circles of a two-dimensional circular lattice in $\mathbb{E}^3$ passing through their centers form a congruence. Let us recall that construction. Denote by C the circle passing through x, $x_{(1)}$ and $x_{(2)}$, and by ν denote the line normal to the plane of the circle and passing through its center (see Fig. 6). The plane bisecting orthogonally the segment $xx_{(2)}$ is the common plane of ν and $\nu_{(-1)}$, and the plane bisecting orthogonally the segment $xx_{(1)}$ is the common plane of ν and $\nu_{(-2)}$, which shows that the family of normals ν is a congruence, called the normal congruence of the circular lattice. This property provides the discrete analogue of the basic relation between normals of a surface in $\mathbb{E}^3$ and the curvature lines.

Due to Proposition 5 we have the discrete counterpart of this characterization of normal congruences as orthogonality of the focal planes "in the mean".

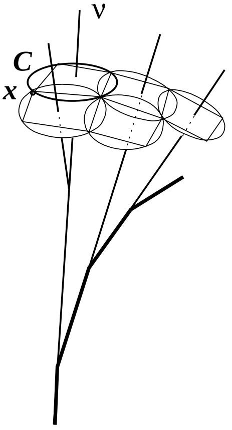

Fig. 6. Normal congruence of a circular lattice.

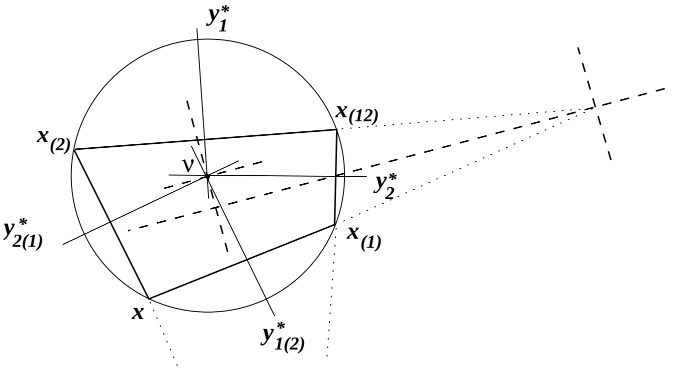

Fig. 7. Discrete normal congruence.

Proposition 6. *If the congruence* ν *with focal lattices* y_1 *and* y_2 *is a normal congruence of the circular lattice* x, *then the pair of orthogonal planes bisecting the angles between the planes* $y^*_{1(-2)}$ *and* y^*_1 *coincides with those bisecting* $y^*_{2(-1)}$ *and* y^*_2.

Proof. Notice that the plane $y^*_{1(-2)}$ containing the lines ν and ν_{-2} is orthogonal to the segment $xx_{(1)}$. The rest of proof follows from similar facts for other planes intersecting at ν and other sides of the quadrilateral and from Proposition 5 (see Fig. 7). □

The property of congruences normal to circular lattices described above suggests the following definition of discrete normal congruences.

Definition 3. A two-dimensional congruence in $\mathbb{E}^3$ is called *normal congruence* if the pairs of planes bisecting the angles between its two corresponding pairs of the consecutive focal planes coincide.

Recall that, within all projective involutions in a pencil of planes in $\mathbb{E}^3$, the reflections are characterized by the property that the fixed planes of the involution are orthogonal. This gives the following, important for further purposes, geometric characterization of the normal congruences

Proposition 7. *Consider a two-dimensional congruence in the Euclidean space* $\mathbb{E}^3$ *whose focal lattices have the tangent planes* y^*_i, $i = 1, 2$. *On each line of the congruence, consider the unique involution in the corresponding pencil of planes mapping* $y^*_{1(-2)}$ *to* y^*_1 *and* $y^*_{2(-1)}$ *to* y^*_2. *The congruence is normal if and only if the fixed planes of the involution are orthogonal*.

There is a natural question if all discrete normal congruences can be constructed from circular lattices. The answer is affirmative but, because we will not use that fact in the sequel, we present here only a geometric sketch of its proof (the interested Reader can find in Appendices A and B the algebraic description of normal congruences and the algebraic proof of the above result). Start with a point $x \in \mathbb{E}^3$ and define its image $x_{(1)}$ in reflection with respect to the plane $y^*_{1(-2)}$ (see Fig. 7). Similarly, define the point $x_{(2)}$ as the image of x in reflection with respect to the plane $y^*_{2(-1)}$. The construction turns out to be compatible, i.e., the image of $x_{(1)}$ in reflection of the plane y^*_2 is the same as the image of $x_{(2)}$ in reflection of the plane y^*_1, due to the normality of the congruence. Moreover, the point $x_{(12)}$ defined in this way is concircular with x, $x_{(1)}$ and $x_{(2)}$. Therefore, given the starting point x_0, one can construct the circular lattice with the congruence being its normal congruence.

4. The discrete Bianchi system

The geometric interpretation of the Bianchi system is given by conjugate nets with equal tangential invariants allowing for a harmonic congruence which is a normal congruence [18]. In such a situation, the involution in the pencil of planes with the base being the harmonic congruence is the orthogonal reflection with respect to the planes of the focal nets. In the previous sections we have defined the integrable discrete analogues of the dual Koenigs net and of the normal congruence. We may therefore expect that, composing both notions in exactly the same way like in the continuous case, we should obtain the discrete Bianchi system together with its geometric interpretation.

Definition 4. The *quadrilateral Bianchi lattice* is a dual Koenigs lattice in $\mathbb{E}^3$ allowing for a harmonic congruence which is a normal congruence.

Remark. We use the name *quadrilateral* Bianchi lattice to distinguish it from the asymptotic Bianchi lattice described in [13]. From now on we study only quadrilateral Bianchi lattices skipping the adjective "quadrilateral".

In studying Bianchi lattices we will often restrict the homogeneous representation of planes to its affine part, i.e., if $\boldsymbol{w}^* = (\vec{\boldsymbol{\omega}}, w^{*4})$ is the tangential homogeneous representation of the plane w^*, its affine part is $\vec{\boldsymbol{\omega}}$. Notice that in this description all parallel planes are indistinguishable.

Proposition 8. *Let x be a dual Koenigs lattice in $\mathbb{E}^3$ with tangential coordinates $\boldsymbol{x}^*$ satisfying* Eq. (9), *denote by $\vec{\boldsymbol{n}}$ its affine part. If the lattice x is a Bianchi lattice, then*

$$\Delta_1\Delta_2(\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{n}} F) = 0. \tag{21}$$

Proof. Let L be a normal congruence harmonic to x. By Proposition 4, we have a unique new dual Koenigs lattice x', harmonic to the congruence and related with x by the corresponding restriction of the fundamental transformation. Denote by $\vec{\boldsymbol{n}}'$ the affine part of the tangential coordinates of x' in the canonical gauge of Theorem 1. Denote by $\vec{\boldsymbol{n}}_\pm$ the affine parts of the tangential coordinates of the fixed planes of the involution on L in the gauge of Corollary 3, i.e.,

$$\vec{\boldsymbol{n}}_\pm = \pm\sqrt{\theta\theta_{(12)}}\vec{\boldsymbol{n}} + \sqrt{\theta_{(1)}\theta_{(2)}}\vec{\boldsymbol{n}}'. \tag{22}$$

By Proposition 7, the congruence L is normal if and only if the vectors $\vec{\boldsymbol{n}}_\pm$ are orthogonal, which gives the following constraint between the affine parts of the tangential coordinates of the Bianchi lattice and of its transform

$$\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{n}} F = \vec{\boldsymbol{n}}' \cdot \vec{\boldsymbol{n}}' F'. \tag{23}$$

Denote by $\vec{\boldsymbol{n}}_i$, $i = 1, 2$, the affine parts of the tangential coordinates of the focal lattices z_i, $i = 1, 2$, of the congruence L in the gauge of Corollary 3, i.e.,

$$\vec{\boldsymbol{n}}_1 = -\theta_{(2)}\theta_{(12)}\frac{\vec{\boldsymbol{n}}}{\phi} - \frac{\vec{\boldsymbol{n}}'}{\phi'} = -\left(\theta\theta_{(1)}\frac{\vec{\boldsymbol{n}}}{\phi} + \frac{\vec{\boldsymbol{n}}'}{\phi'}\right)_{(2)}, \tag{24}$$

$$\vec{\boldsymbol{n}}_2 = \theta_{(1)}\theta_{(12)}\frac{\vec{\boldsymbol{n}}}{\phi} - \frac{\vec{\boldsymbol{n}}'}{\phi'} = \left(\theta\theta_{(2)}\frac{\vec{\boldsymbol{n}}}{\phi} - \frac{\vec{\boldsymbol{n}}'}{\phi'}\right)_{(1)}. \tag{25}$$

Then Eq. (23) implies that

$$\frac{\vec{\boldsymbol{n}}_2 \cdot \vec{\boldsymbol{n}}_2}{\theta_{(1)}\theta_{(12)}} + \frac{\vec{\boldsymbol{n}}_1 \cdot \vec{\boldsymbol{n}}_1}{\theta_{(2)}\theta_{(12)}} = \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{n}} F \tag{26}$$

and

$$\Delta_1\left(\frac{\vec{\boldsymbol{n}}_{2(-1)} \cdot \vec{\boldsymbol{n}}_{2(-1)}}{\theta\theta_{(2)}}\right) = 0, \qquad \Delta_2\left(\frac{\vec{\boldsymbol{n}}_{1(-2)} \cdot \vec{\boldsymbol{n}}_{1(-2)}}{\theta\theta_{(1)}}\right) = 0, \tag{27}$$

which give the constraint (21). □

Corollary 4. *The condition* (21) *implies that*

$$\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{n}} F = U_1(m_1) + U_2(m_2), \tag{28}$$

where U_1 and U_2 are functions of the single variables m_1 and m_2, respectively. In the above notation

$$\vec{\boldsymbol{n}}_{1(-2)} \cdot \vec{\boldsymbol{n}}_{1(-2)} = \theta\theta_{(1)}(U_1(m_1) + \lambda), \qquad \vec{\boldsymbol{n}}_{2(-1)} \cdot \vec{\boldsymbol{n}}_{2(-1)} = \theta\theta_{(2)}(U_2(m_2) - \lambda), \tag{29}$$

where λ is a constant.

Corollary 5. *Because the congruence L is also harmonic to the new dual Koenigs lattice x', then the lattice is also a Bianchi lattice.*

Our last step is to show that the condition described in Proposition 8 is also sufficient to characterize the Bianchi lattices among the dual Koenigs lattices. In order to make clear the geometric content of the forthcoming calculations, let us first draw some consequences of the previous considerations.

Let x be a dual Koenigs lattice and let L be a harmonic congruence conjugated to the lattice. Denote by $\boldsymbol{z}_1^*$ and $\boldsymbol{z}_2^*$ the tangential coordinates of the focal lattices of L in the gauge of Corollary 3. The plane w^* with homogeneous coordinates

$$\boldsymbol{w}^* = -\boldsymbol{z}^*_{1(-2)} - \boldsymbol{z}^*_{2(-1)} \tag{30}$$

contains the line L, while Eqs. (18) and (19) imply that $\boldsymbol{w}^*$ satisfies the linear system

$$\Delta_1 \boldsymbol{w}^* = \theta_{(1)} \boldsymbol{x}^*_{(1)} - (2F\theta_{(1)} - \theta)\boldsymbol{x}^*, \tag{31}$$

$$\Delta_2 \boldsymbol{w}^* = -\theta_{(2)} \boldsymbol{x}^*_{(2)} + (2F\theta_{(2)} - \theta)\boldsymbol{x}^* \tag{32}$$

and, by Eqs. (15) and (16), the tangential coordinates of the corresponding Koenigs transform x' read

$$\boldsymbol{x}'^* = \frac{1}{2}\left[\left(\frac{1}{\theta_{(1)}} - \frac{1}{\theta_{(2)}}\right)\theta \boldsymbol{x}^* + \left(\frac{1}{\theta_{(1)}} + \frac{1}{\theta_{(2)}}\right)\boldsymbol{w}^*\right]. \tag{33}$$

If, moreover, x is a Bianchi lattice and the congruence L is normal, then the affine part $\vec{\boldsymbol{\omega}} = -\vec{\boldsymbol{n}}_{1(-2)} - \vec{\boldsymbol{n}}_{2(-1)}$ of $\boldsymbol{w}^*$ is subjected also to the following constraints

$$\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{n}} = \theta_{(1)}(U_1 + \lambda) - \theta_{(2)}(U_2 - \lambda), \tag{34}$$

$$\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{\omega}} = 2((U_1 + \lambda)\theta\theta_{(1)} + (U_2 - \lambda)\theta\theta_{(2)}) - \theta^2 \frac{U_1 + U_2}{F}. \tag{35}$$

Proposition 9. *Let x be a dual Koenigs lattice in $\mathbb{E}^3$ with tangential coordinates $\boldsymbol{x}^*$ satisfying* Eq. (9)*. If its affine part $\vec{\boldsymbol{n}}$ satisfies* Eq. (21) *then the lattice x is a Bianchi lattice.*

Proof. The idea is to construct a normal congruence harmonic to the given dual Koenigs lattice subjected to condition (21). Let $\boldsymbol{x}^* = (\vec{\boldsymbol{n}}, x^{*4})$ be the homogeneous tangential coordinates of such a lattice in the canonical gauge. Fix the functions U_1, U_2 in Eq. (28) and fix a parameter λ (it plays the role of spectral parameter in soliton theory).

Given (from previous steps of the construction or as initial data) the plane w^* with homogeneous coordinates $\boldsymbol{w}^* = (\vec{\boldsymbol{\omega}}, w^{*4})$ and the scalar θ. Define the planes $z^*_{2(-1)}$ and $z^*_{1(-2)}$ with homogeneous coordinates $\boldsymbol{z}^*_{2(-1)} = (\vec{\boldsymbol{n}}_{2(-1)}, z^{*4}_{2(-1)})$ and $\boldsymbol{z}^*_{1(-2)} = (\vec{\boldsymbol{n}}_{1(-2)}, z^{*4}_{1(-2)})$ by equations

$$\boldsymbol{z}^*_{1(-2)} = -\tfrac{1}{2}(\theta \boldsymbol{x}^* + \boldsymbol{w}^*), \tag{36}$$

$$\boldsymbol{z}^*_{2(-1)} = \tfrac{1}{2}(\theta \boldsymbol{x}^* - \boldsymbol{w}^*). \tag{37}$$

Define $\theta_{(1)}$ and $\theta_{(2)}$ by Eqs. (29), then the affine part $\vec{\boldsymbol{\omega}}$ of $\boldsymbol{w}^*$ satisfies Eqs. (34) and (35). In the last step of the construction we find $\boldsymbol{w}^*_{(1)}$ and $\boldsymbol{w}^*_{(2)}$ from Eqs. (31) and (32).

By Eqs. (34) and (35) the construction is compatible and, moreover, the function θ satisfies the discrete Moutard equation (10). Then Eqs. (36) and (37) and Eqs. (31) and (32) imply that $\boldsymbol{z}^*_1$ and $\boldsymbol{z}^*_2$ satisfy Eqs. (17)–(19), i.e., the lattices z_1 and z_2 are focal lattices of a congruence L harmonic to x.

Because the vectors $\vec{\boldsymbol{n}}_{1(-2)}/\sqrt{\theta\theta_{(1)}}$ and $\vec{\boldsymbol{n}}_1/\sqrt{\theta_{(2)}\theta_{(12)}}$ are of equal length, then the planes bisecting $z^*_{1(-2)}$ and z^*_1 have the normal vectors

$$\vec{\boldsymbol{b}}_{1\mp} = \frac{\vec{\boldsymbol{n}}_{1(-2)}}{\sqrt{\theta\theta_{(1)}}} \pm \frac{\vec{\boldsymbol{n}}_1}{\sqrt{\theta_{(2)}\theta_{(12)}}},$$

similarly, the planes bisecting $z^*_{2(-1)}$ and z^*_2 have the normal vectors

$$\vec{\boldsymbol{b}}_{2\pm} = \frac{\vec{\boldsymbol{n}}_{2(-1)}}{\sqrt{\theta\theta_{(2)}}} \pm \frac{\vec{\boldsymbol{n}}_2}{\sqrt{\theta_{(1)}\theta_{(12)}}}.$$

Eqs. (28), (34) and (35) imply

$$\vec{\boldsymbol{b}}_{1-} \cdot \vec{\boldsymbol{b}}_{2+} = \vec{\boldsymbol{b}}_{1+} \cdot \vec{\boldsymbol{b}}_{2-} = 0,$$

which shows that the congruence L is normal. □

Corollary 6. *Once the normal congruence L is found then the tangential coordinates of the corresponding new Bianchi lattice x' are given by* (33).

Remark. The vectors $\vec{\boldsymbol{b}}_{i\pm}$ are proportional to the vectors $\vec{\boldsymbol{n}}_{\pm}$ considered in the proof of Proposition 8.

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Appendix A. Algebraic description of normal congruences

The goal of appendix is to present the theory of discrete normal congruences using the more algebraic language of difference equations.

A.1. Quadrilateral lattices and congruences in the affine gauge

Let us first express some facts from the theory of quadrilateral lattices and discrete congruences in the language of affine geometry which will be used in the algebraic description of the discrete normal congruences.

Restricting the ambient space from the projective space $\mathbb{P}^3$ to the corresponding affine space $\mathbb{R}^3$, we change from the homogeneous coordinates $\boldsymbol{x}$ to the nonhomogeneous ones $\vec{\boldsymbol{x}}$, i.e. $[\boldsymbol{x}] = [(\vec{\boldsymbol{x}}, 1)]$. Then the Laplace system takes the following form [8]

$$\vec{\boldsymbol{x}}_{(12)} = A_{(1)}\vec{\boldsymbol{x}}_{(1)} + B_{(2)}\vec{\boldsymbol{x}}_{(2)} + (1 - A_{(1)} - B_{(2)})\vec{\boldsymbol{x}}. \tag{A.1}$$

Define the Lamé coefficients H and G as "logarithmic potentials"

$$A = \frac{H_{(2)}}{H}, \qquad B = \frac{G_{(1)}}{G} \tag{A.2}$$

and introduce the suitably scaled tangent vectors $\vec{\boldsymbol{X}}$, $\vec{\boldsymbol{Y}}$,

$$\Delta_1\vec{\boldsymbol{x}} = H_{(1)}\vec{\boldsymbol{X}}, \qquad \Delta_2\vec{\boldsymbol{x}} = G_{(2)}\vec{\boldsymbol{Y}}. \tag{A.3}$$

Corollary A.1. *The Lamé coefficients are not unique*; *H is given up to the multiplication by a function of the single argument m_1, while G is given up to the multiplication by a function of the single argument m_2*

In terms of such vectors, the affine Laplace equation (A.1) changes into the system of two equations of the first order

$$\Delta_2\vec{\boldsymbol{X}} = Q_{(2)}\vec{\boldsymbol{Y}}, \qquad \Delta_1\vec{\boldsymbol{Y}} = P_{(1)}\vec{\boldsymbol{X}}, \tag{A.4}$$

where the proportionality factors Q and P, called the rotation coefficients, are given by

$$Q = \frac{\Delta_1 G}{H_{(1)}}, \qquad P = \frac{\Delta_2 H}{G_{(2)}}. \tag{A.5}$$

Denote by $\vec{\boldsymbol{y}}_i$, $i = 1, 2$, the affine representants of the focal lattices of a congruence. The coefficients of their Laplace equations

$$\vec{\boldsymbol{y}}_{i(12)} = A^i_{(1)}\vec{\boldsymbol{y}}_{i(1)} + B^i_{(2)}\vec{\boldsymbol{y}}_{i(2)} + (1 - A^i_{(1)} - B^i_{(2)})\vec{\boldsymbol{y}}_i, \quad i = 1, 2,$$

enter into the formulas connecting the lattices as follows [8,16]

$$\vec{\boldsymbol{y}}_2 - \vec{\boldsymbol{y}}_1 = -\frac{1}{B^1 - 1}\Delta_1\vec{\boldsymbol{y}}_1 = \frac{1}{A^2 - 1}\Delta_2\vec{\boldsymbol{y}}_2. \tag{A.6}$$

Eq. (A.6) leads directly to the following result.

Proposition A.1. *Consider a two-dimensional congruence whose focal lattices are represented in the affine gauge by $\vec{\boldsymbol{y}}_i$, $i = 1, 2$, and denote by H^i, G^i, their Lamé coefficients. Define the vectors $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{Y}}$ as follows*

$$\tilde{\boldsymbol{X}} = \frac{\vec{\boldsymbol{y}}_2}{H^2}, \qquad \tilde{\boldsymbol{Y}} = \frac{\vec{\boldsymbol{y}}_1}{G^1}. \tag{A.7}$$

Then these vectors satisfy the equations

$$\Delta_2\tilde{\boldsymbol{X}} = G^1\Delta_2\left(\frac{1}{H^2}\right)\tilde{\boldsymbol{Y}}, \qquad \Delta_1\tilde{\boldsymbol{Y}} = H^2\Delta_1\left(\frac{1}{G^1}\right)\tilde{\boldsymbol{X}}. \tag{A.8}$$

Remark. Notice that the pair $(1/H^2, 1/G^1)$ satisfies the same linear problem (A.8) as the pair $(\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}})$.

Corollary A.2. *Consider a pair (X^0, Y^0) of scalar solutions of the linear problem* (A.4), *then*

$$\vec{\boldsymbol{y}}_1 = \frac{\vec{\boldsymbol{Y}}}{Y^0}, \qquad \vec{\boldsymbol{y}}_2 = \frac{\vec{\boldsymbol{X}}}{X^0},$$

are affine coordinates of the focal lattices of a congruence. Moreover, the Lamé coefficients of these lattices can be chosen in such a way that

$$H^2 = \frac{1}{X^0}, \qquad G^1 = \frac{1}{Y^0}.$$

We represent planes in $\mathbb{R}^3$ by dual vectors $\vec{\boldsymbol{x}}^*$ such that $[\boldsymbol{x}^*] = [(\vec{\boldsymbol{x}}^*, -1)]$. Then the equation of the plane represented by $\vec{\boldsymbol{x}}^*$ and passing through the point represented by $\vec{\boldsymbol{x}}$ is normalized to

$$\langle \vec{\boldsymbol{x}}^*, \vec{\boldsymbol{x}} \rangle = 1.$$

Remark. Notice that the dual analogue of the points of the plane at infinity is the set of all planes passing through the origin.

In the transition from the homogeneous tangential coordinates of quadrilateral lattices to the nonhomogeneous ones, the Laplace equation (7) is replaced by its affine form (A.1), but with the coefficients A^* and B^*. Therefore, also other algebraic considerations, from Eqs. (A.3)–(A.5), apply to the affine tangential coordinates of quadrilateral lattices. In this way one defines the dual Lamé coefficients H^* and G^*, the dual normalized tangent vectors $\vec{\boldsymbol{X}}^*$, $\vec{\boldsymbol{Y}}^*$, and the dual rotation coefficients P^* and Q^*.

The dual version of the connection formulas (A.6) can be found from the requirement that the four planes represented by $\vec{\boldsymbol{y}}_1^*$, $\vec{\boldsymbol{y}}_{1(-2)}^*$, $\vec{\boldsymbol{y}}_2^*$ and $\vec{\boldsymbol{y}}_{2(-1)}^*$ intersect along one line, exactly in the same way like Eq. (A.6) were found from requirement that the four points represented by $\vec{\boldsymbol{y}}_1$, $\vec{\boldsymbol{y}}_{1(1)}$, $\vec{\boldsymbol{y}}_2$ and $\vec{\boldsymbol{y}}_{2(2)}$ belong to one line. Such a derivation, which gives the same result as applying the duality principle to equations (A.6) (and taking into account Corollary 1), leads to the following connection formulas

$$\vec{\boldsymbol{y}}_{1(-2)}^* - \vec{\boldsymbol{y}}_{2(-1)}^* = -\frac{1}{B_{(-1)}^{*2} - 1} \Delta_1 \vec{\boldsymbol{y}}_{2(-1)}^* = \frac{1}{A_{(-2)}^{*1} - 1} \Delta_2 \vec{\boldsymbol{y}}_{1(-2)}^*, \tag{A.9}$$

which imply the following dual versions of Proposition A.1 and Corollary A.2.

Proposition A.2. *Consider a two-dimensional congruence in $\mathbb{P}^3$ and its focal lattices whose tangent planes are represented in the affine gauge by $\vec{\boldsymbol{y}}_i^*$, $i = 1, 2$. Denote by H^{*i}, G^{*i}, their Lamé coefficients and define the dual vectors $\tilde{\boldsymbol{X}}^*$ and $\tilde{\boldsymbol{Y}}^*$ as follows*

$$\tilde{\boldsymbol{X}}^* = \left(\frac{\vec{\boldsymbol{y}}_1^*}{H^{*1}} \right)_{(-2)}, \qquad \tilde{\boldsymbol{Y}}^* = \left(\frac{\vec{\boldsymbol{y}}_2^*}{G^{*2}} \right)_{(1)}. \tag{A.10}$$

Then these vectors satisfy the equations

$$\Delta_2 \tilde{\boldsymbol{X}}^* = G_{(-1)}^{*2} \Delta_2 \left(\frac{1}{H_{(-2)}^{*1}} \right) \tilde{\boldsymbol{Y}}^*, \qquad \Delta_1 \tilde{\boldsymbol{Y}}^* = H_{(-2)}^{*1} \Delta_1 \left(\frac{1}{G_{(-1)}^{*2}} \right) \tilde{\boldsymbol{X}}^*. \tag{A.11}$$

Remark. Notice that the pair $(1/H_{(-2)}^{*1}, 1/G_{(-1)}^{*2})$ satisfies the same linear problem (A.8) as the pair $(\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}})$.

Corollary A.3. *Given a pair* (X^0, Y^0) *of scalar solutions of the linear problem* (A.4), *then*

$$\vec{y}_1^* = \left(\frac{\vec{X}}{X^0}\right)_{(2)}, \qquad \vec{y}_2^* = \left(\frac{\vec{Y}}{Y^0}\right)_{(1)}$$

are affine tangential coordinates of the focal lattices of a congruence. Moreover, the Lamé coefficients of the lattices $\vec{y}_1^*$ *and* $\vec{y}_2^*$ *can be chosen in such a way that*

$$G^{*2} = \frac{1}{Y^0_{(1)}}, \qquad H^{*1} = \frac{1}{X^0_{(2)}}.$$

A.2. Circular lattices and normal congruences

In the Euclidean space $\mathbb{E}^3$ we identify covectors with vectors via the scalar product and we represent planes using their normals. The affine representation of a plane is given by $\vec{x}^* = \vec{\eta}/w$, where $\vec{\eta}$ is the unit outer (with respect to the origin) normal vector to the plane and w is the distance of the plane from the origin.

It can be shown [12] that a quadrilateral lattice is circular (see Section 3) if and only if its normalized tangent vectors satisfy the constraint

$$\vec{X} \cdot \vec{Y}_{(1)} + \vec{Y} \cdot \vec{X}_{(2)} = 0, \tag{A.12}$$

which, in the continuous limit, gives the orthogonality of the curvature lines. Other two convenient characterizations of the circular lattice are as follows (see [9,23] for more details): the quadrilateral lattice x is circular if and only if the scalars

$$X^\circ := \tfrac{1}{2}(\vec{x}_{(1)} + \vec{x}) \cdot \vec{X}, \qquad Y^\circ := \tfrac{1}{2}(\vec{x}_{(2)} + \vec{x}) \cdot \vec{Y}, \tag{A.13}$$

solve the linear system (A.4) or, equivalently, if the function $|\vec{x}|^2 = \vec{x} \cdot \vec{x}$ satisfies the affine Laplace equation (A.1) of $\vec{x}$.

The algebraic version of the new geometric characterization of circular lattices described in Proposition 5 (we consider here also the second pair of bisectrices) is contained in the following.

Proposition A.3. *For* $\vec{X}$ *and* $\vec{Y}$ *satisfying the linear problem* (A.4), *the condition* (A.12) *is equivalent to the constraint*

$$\left(\frac{\vec{X}}{|\vec{X}|} + \frac{\vec{X}_{(2)}}{|\vec{X}_{(2)}|}\right)\left(\frac{\vec{Y}}{|\vec{Y}|} + \frac{\vec{Y}_{(1)}}{|\vec{Y}_{(1)}|}\right) = \left(\frac{\vec{X}}{|\vec{X}|} - \frac{\vec{X}_{(2)}}{|\vec{X}_{(2)}|}\right)\left(\frac{\vec{Y}}{|\vec{Y}|} - \frac{\vec{Y}_{(1)}}{|\vec{Y}_{(1)}|}\right) = 0. \tag{A.14}$$

Proof. Eq. (A.14) are equivalent to equations

$$\frac{\vec{X}}{|\vec{X}|}\frac{\vec{Y}}{|\vec{Y}|} + \frac{\vec{X}_{(2)}}{|\vec{X}_{(2)}|}\frac{\vec{Y}_{(1)}}{|\vec{Y}_{(1)}|} = \frac{\vec{X}}{|\vec{X}|}\frac{\vec{Y}_{(1)}}{|\vec{Y}_{(1)}|} + \frac{\vec{X}_{(2)}}{|\vec{X}_{(2)}|}\frac{\vec{Y}}{|\vec{Y}|} = 0. \tag{A.15}$$

($\Rightarrow$) Eq. (A.12) and the linear problem (A.4) imply [12], that

$$\frac{|\vec{X}_{(2)}|}{|\vec{X}|} = \frac{|\vec{Y}_{(1)}|}{|\vec{Y}|} = \sqrt{1 - Q_{(2)}P_{(1)}}, \tag{A.16}$$

which together with the following consequence of (A.4)

$$\vec{X}_{(2)} \cdot \vec{Y}_{(1)} = (1 + P_{(1)}Q_{(2)})\vec{X} \cdot \vec{Y} + P_{(1)}|\vec{X}|^2 + Q_{(2)}|\vec{Y}|^2, \tag{A.17}$$

lead to Eq. (A.15).

(⇐) Eq. (A.15) and the linear problem (A.4) imply (A.16) which gives condition (A.12). □

Remark. Eq. (A.15) express the basic property of circular quadrilaterals, mentioned in a proof of Proposition 5, in the form independent of the convexity or not of the quadrilateral (see [12]).

The following proposition gives the tangential coordinates of focal lattices of the normal congruence of the circular lattice x in terms of point coordinates of the lattice.

Proposition A.4. *Given a two-dimensional circular lattice $x : \mathbb{Z}^2 \to \mathbb{E}^3$, then the tangent planes of the focal lattices of the normal congruence ν of x are represented in the affine gauge by their normals*

$$\vec{y}_1^* = \left(\frac{\Delta_1 \vec{x}}{\Delta_1 |\vec{x}|^2} \right)_{(2)} = \left(\frac{\vec{X}}{X^\circ} \right)_{(2)}, \qquad \vec{y}_2^* = \left(\frac{\Delta_2 \vec{x}}{\Delta_2 |\vec{x}|^2} \right)_{(1)} = \left(\frac{\vec{Y}}{Y^\circ} \right)_{(1)}. \tag{A.18}$$

Moreover, the dual Lamé coefficients of the lattices $\vec{y}_1^*$ and $\vec{y}_2^*$ can be chosen in such a way that

$$G^{*2} = \frac{1}{Y^\circ_{(1)}}, \qquad H^{*1} = \frac{1}{X^\circ_{(2)}}. \tag{A.19}$$

Proof. The plane y_1^* contains the lines ν and $\nu_{(2)}$, therefore its normal must be in the direction of the line passing through the points $x_{(2)}$ and $x_{(12)}$; i.e. it must be proportional to $\Delta_1 \vec{x}_{(2)}$. The normalization factor can be found from the condition that the plane passes through the middle-point of the segment $x_{(2)}x_{(12)}$. Similarly, we find the expression for $\vec{y}_2^*$. The rest of the proposition follows from Corollary A.3. □

Let us give the algebraic characterization, modeled on Proposition A.4, of the normal congruences.

Proposition A.5. *Consider a two-dimensional congruence in the Euclidean space, with the focal lattices represented in the affine gauge by their normals $\vec{y}_i^*$, $i = 1, 2$, and denote by H^{*i}, G^{*i}, their dual Lamé coefficients. The congruence is normal if and only if*

$$\left(\frac{\vec{y}_1^*}{H^{*1}} \right)_{(-2)} \left(\frac{\vec{y}_2^*}{G^{*2}} \right) + \left(\frac{\vec{y}_2^*}{G^{*2}} \right)_{(-1)} \left(\frac{\vec{y}_1^*}{H^{*1}} \right) = 0. \tag{A.20}$$

Proof. Define the vectors $\vec{X}$ and $\vec{Y}$ by Eq. (A.10); then the condition (A.20) is transformed into Eq. (A.12). By Proposition A.2 the vectors $\vec{X}$, $\vec{Y}$ satisfy the linear problem of the form

(A.4), with the rotation coefficients P, Q given by

$$P_{(-1)} = H^{*1}_{(-2)}\Delta_1\left(\frac{1}{G^{*2}_{(-1)}}\right), \qquad Q_{(-2)} = G^{*2}_{(-1)}\Delta_2\left(\frac{1}{H^{*1}_{(-2)}}\right). \tag{A.21}$$

The vectors

$$\vec{\boldsymbol{B}}_{1\pm} = \frac{\vec{\boldsymbol{X}}}{|\vec{\boldsymbol{X}}|} \pm \frac{\vec{\boldsymbol{X}}_{(2)}}{|\vec{\boldsymbol{X}}_{(2)}|}$$

are normal vectors of two orthogonal planes bisecting $y^*_{1(-2)}$ and y^*_1, similarly the vectors

$$\vec{\boldsymbol{B}}_{2\pm} = \frac{\vec{\boldsymbol{Y}}}{|\vec{\boldsymbol{Y}}|} \pm \frac{\vec{\boldsymbol{Y}}_{(1)}}{|\vec{\boldsymbol{Y}}_{(1)}|}$$

are normal vector of the planes bisecting $y^*_{2(-1)}$ and y^*_2. The rest of the proof follows from Proposition A.3. □

Proposition A.6. *Given a discrete normal congruence* ν, *then there exists a circular lattice* x *such that* ν *is the normal congruence of* x.

Proof. Define the vectors $\vec{\boldsymbol{X}}$, $\vec{\boldsymbol{Y}}$ and their rotation coefficients P, Q like in the proof of Proposition A.5. By the remark after Proposition A.2 the functions X°, Y°, defined by Eq. (A.19), satisfy the same linear problem as $\vec{\boldsymbol{X}}$, $\vec{\boldsymbol{Y}}$.

Consider the following linear system for $\vec{\boldsymbol{x}}$

$$\vec{\boldsymbol{x}}_{(1)} = \vec{\boldsymbol{x}} + \frac{2(X^\circ - \vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{X}})}{|\vec{\boldsymbol{X}}|^2}\vec{\boldsymbol{X}}, \tag{A.22}$$

$$\vec{\boldsymbol{x}}_{(2)} = \vec{\boldsymbol{x}} + \frac{2(Y^\circ - \vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{Y}})}{|\vec{\boldsymbol{Y}}|^2}\vec{\boldsymbol{Y}}, \tag{A.23}$$

whose geometric interpretation was described at the end of Section 3. Its compatibility is asserted by Eq. (A.4), (A.12) and their consequence (A.16).

The functions H and G defined by

$$H_{(1)} = \frac{2(X^\circ - \vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{X}})}{|\vec{\boldsymbol{X}}|^2}, \qquad G_{(2)} = \frac{2(Y^\circ - \vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{Y}})}{|\vec{\boldsymbol{Y}}|^2},$$

satisfy Eq. (A.5) and are connected with $\vec{\boldsymbol{x}}$ and the pair $(\vec{\boldsymbol{X}}, \vec{\boldsymbol{Y}})$ by Eq. (A.3). Therefore, the lattice represented by $\vec{\boldsymbol{x}}$ is a quadrilateral lattice with the normalized tangent vectors $\vec{\boldsymbol{X}}$ and $\vec{\boldsymbol{Y}}$, and the Lamé coefficients H and G. Because $\vec{\boldsymbol{X}}$ and $\vec{\boldsymbol{Y}}$ satisfy the constraint (A.12), the lattice is circular.

Eqs. (A.22) and (A.23) imply that the circular lattice x and the focal lattices y_i of the congruence are connected by Eq. (A.18); which means just that the focal planes bisect orthogonally corresponding segments of the lattice, i.e. the congruence ν is the normal congruence of x. □

Remark. If there exists one such circular lattice, then there exist an infinity of such circular lattices labeled by the position of an initial point of the construction.

Appendix B. The Darboux transformation for the discrete Bianchi system

The following results can be checked by direct (but tedious) calculation and actually are modification of Theorem 7 of [13].

Lemma B.1. *Given* $\vec{\boldsymbol{n}} : \mathbb{Z}^2 \to \mathbb{E}^3$ *satisfying the discrete Bianchi system* (5), *define* $\vec{\boldsymbol{\eta}}_0 = F\vec{\boldsymbol{n}}$ *and let* $\vec{\boldsymbol{\eta}}_1$ *and* $\vec{\boldsymbol{\eta}}_2$ *be unit vectors orthogonal to* $\vec{\boldsymbol{\eta}}_0$ *and to each other. If* $p_A^B, q_A^B, A, B = 0, 1, 2$, *are the functions defined by the unique decompositions*

$$\vec{\boldsymbol{\eta}}_A = \sum_{B=0}^{2} p_A^B \vec{\boldsymbol{\eta}}_{B(1)}, \qquad \vec{\boldsymbol{\eta}}_A = \sum_{B=0}^{2} q_A^B \vec{\boldsymbol{\eta}}_{B(2)},$$

and if

$$R(m_1, m_2) = U_1(m_1) + U_2(m_2), \quad a(m_1) = U_1(m_1) + \lambda, \qquad b(m_2) = U_2(m_2) - \lambda,$$

then:

(i) *the linear system*

$$\Theta_{(i)} = M_i \Theta, \quad i = 1, 2, \tag{B.1}$$

where $\Theta = (\theta, \theta', \theta'', y^1, y^2)^{\mathrm{T}}$, and

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \dfrac{R_{(1)}(p_0^0/F) - b}{a_{(1)}} & \dfrac{bF - R_{(1)}(((R+b)/R)p_0^0 - (1/F_{(1)}))}{a_{(1)}} & \dfrac{b}{a_{(1)}}\left(F - \dfrac{R_{(1)}}{R} p_0^0\right) & \dfrac{R_{(1)}}{a_{(1)}} p_1^0 & \dfrac{R_{(1)}}{a_{(1)}} p_2^0 \\ -1 & F & F & 0 & 0 \\ \dfrac{p_0^1}{F} & -\dfrac{R+b}{R} p_0^1 & -\dfrac{b}{R} p_0^1 & p_1^1 & p_2^1 \\ \dfrac{p_0^2}{F} & -\dfrac{R+b}{R} p_0^2 & -\dfrac{b}{R} p_0^2 & p_1^2 & p_2^2 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -1 & F & F & 0 & 0 \\ \dfrac{R_{(2)}(q_0^0/F) - a}{b_{(2)}} & \dfrac{a}{b_{(2)}}\left(F - \dfrac{R_{(2)}}{R} q_0^0\right) & \dfrac{aF - R_{(2)}(((R+a)/R)q_0^0 - (1/F_{(2)}))}{b_{(2)}} & -\dfrac{R_{(2)}}{b_{(2)}} q_1^0 & -\dfrac{R_{(2)}}{b_{(2)}} q_2^0 \\ -\dfrac{q_0^1}{F} & \dfrac{a}{R} q_0^1 & \dfrac{a+R}{R} q_0^1 & q_1^1 & q_2^1 \\ -\dfrac{q_0^2}{F} & \dfrac{a}{R} q_0^2 & \dfrac{a+R}{R} q_0^2 & q_1^2 & q_2^2 \end{pmatrix}$$

is compatible;

(ii) *the function*

$$I = (y^1)^2 + (y^2)^2 + \frac{R}{F}\theta^2 + \frac{F}{R}(b\theta'' - a\theta')^2 - 2\theta(a\theta' + b\theta'')$$

is a first integral of the system;

(iii) *the function* $\vec{\boldsymbol{\omega}} = \sum_{A=0}^{2} y^A \vec{\boldsymbol{\eta}}_A$ *with*

$$y^0 = \frac{a\theta' - b\theta''}{R}$$

satisfies the linear system (31) and (32) *with* $\vec{\boldsymbol{n}}$ *in place of* $\boldsymbol{x}^*$.

Corollary B.1. *Notice that the solution* θ *of the linear system* (B.1) *is a solution of the discrete Moutard equation* (10), *while* $\theta' = \theta_{(1)}$ *and* $\theta'' = \theta_{(2)}$.

Theorem B.1. *Given* $\vec{\boldsymbol{n}} : \mathbb{Z}^2 \to \mathbb{E}^3$ *satisfying the discrete Bianchi system* (5), *let* θ *and* $\vec{\boldsymbol{\omega}}$ *be obtained from the solution of the linear system* (B.1) *subjected to the admissible constraint* $I = 0$. *Then* $\vec{\boldsymbol{\omega}}$ *satisfies* Eqs. (34) and (35) *and* $\vec{\boldsymbol{n}}'$, *given by* Eq. (33) *with* $\vec{\boldsymbol{n}}$ *and* $\vec{\boldsymbol{\omega}}$ *in place of* $\boldsymbol{x}^*$ *and* $\boldsymbol{w}^*$, *is a new solution of the discrete Bianchi system* (5).

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